λ -TYPE PHASE TRANSITION FOR A WEAKLY INTERACTING BOSE GAS

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Abstract

A mechanism generating a λ-type behavior in the specific heat of a Bose gas near the critical temperature T_c is discussed. It is shown to work for a general class of quasiparticle spectra, and especially, for the Bogoliubov's model of a weakly interacting Bose gas, where a temperature-dependent quasi-particle spectrum is obtained.

1 Introduction

The first experimental measurement of the low temperature specific heat of He^4 was made by Keesom in 1927-8 [1] who became impressed by its λ shape and by its singular behavior at the critical temperature where superfluid behavior started to manifest. In his book on Statistical Mechanics, [2] Feynman refers to the λ point behavior as one of the unsolved problems of superfluidity theory, and expresses his view that *perhaps part of the explanation of the lambda transition involves Bose condensation* since the $c_v(T)$ -curve for an ideal Bose gas shows a peaked form at the critical temperature, although it has not a divergent behavior. We are not aware about any satisfactory model explaining this behavior, except the one presented in the paper by D.M.Ceperley [3] in which it is simulated the boson system by means of the Monte Carlo techniques using path integrals. In that paper, a good agreement is obtained among the simulation and the experimental measurements of the He^4 specific heat.

In our present letter we want to discuss a mechanism for generating a λ-type behavior in the specific heat of a Bose gas. We consider at first some models for the phase transition of symmetry restoration in temperature relativistic systems with spontaneous symmetry breaking, which exhibit a λ-type behavior for c_v, and then we study the similar problem in the non-relativistic case. We show that for the *weakly interacting Bose gas* one can find such a divergent behavior of the specific heat as being produced by the condensate. Obviously, such a system is far from being a satisfactory model for the superfluid, in which the interactions cannot be considered weak at all. However, as the mechanism is essentially an infrared (long wavelength) effect, it may provide insights in understanding what happen in a more exact model of superfluidity. Also, the behavior of the specific heat for the weakly interacting Bose gas may become interesting in connection with the recent experimentally observed Bose condensate, although such problem would require to consider the particles in an external field, provided by the magnetic trap.

neous symmetry breaking (SSB). For the latter, one can find that a singular behavior of the specific heat occurs at the symmetry restoration temperature, due to the temperature-dependent mass, proportional to the symmetry-breaking parameter. We prove that a similar phenomenon occurs for boson systems whose quasiparticle spectra have some specific non-vanishing temperature-dependent spectra in the infrared limit.

We will take either the simple model of the scalar field or the Abelian Higgs model [4], [5]. One can write the effective potential as

$$V(\xi) = \frac{\lambda\xi}{4} - \frac{a^2\xi^2}{2} + V(T,\xi), \tag{1}$$

where $V(\xi)$ is the sum of tadpole diagrams. The spectrum is $\epsilon(p)_i = c\sqrt{(p^2 + M_i^2 c^2}$, where $i = 1$ for the scalar and $i = 1, 2, 3, 4$ for the multiplet in the scalar-vector Higgs model where, $M_1 = \lambda\xi$, and $M_{2,3,4} = g\xi/2$ and g is the scalar-vector coupling constant. In the high temperature limit is $V(T,\xi) = \alpha T^2 \xi^2/2$, the extremum of $V(\xi)$ leading for $T < T_c$ to a dependence $M(T)_i = \kappa_i\sqrt{T_c^2 - T^2}$, T_c being the critical temperature for symmetry restoration. The infrared contribution to the thermodynamic potential in the limit $T \gg M(T)_i$ is $\Delta\Omega \simeq \frac{\sum c^3 M_i^3 T}{12\pi h^3}$. One obtains then a specific heat

$$\Delta c_v = -T\frac{\partial^2\Omega}{\partial T^2} \simeq \sum c^3\kappa_i^3 \frac{T^4}{2\pi h^3 \sqrt{T_c^2 - T^2}}, \tag{2}$$

which diverges for $T \to T_c$, showing a λ-type behavior.

In general, we may argue that any quasi-particle spectrum of the Bose gas whose infrared limit behaves as $\varepsilon(\mathbf{p}) = a\sqrt{\delta^2 + p^2}$, where $\delta = b\sqrt{1 - (T/T_c)^\gamma}$, and $\gamma > 1$, leads to a λ-type behavior of c_v. One can obtain that result from the infrared term in the calculations made in [4]. But also, by taking T close enough to T_c to make δ arbitrary small, one can choose some small momentum $\eta \gg \delta$. Then the integral over p in the thermodynamic potential $\Omega = \frac{V}{\beta h^3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln[1 - \exp(-\beta\varepsilon(\mathbf{p}))]$ can be decomposed in the sum $\int_0^\eta + \int_\eta^\infty$, from which the infrared contribution as $\Delta\Omega = \delta^3 T/12\pi h^3$ is obtained, leading to the λ-type behavior for $c_v = -T\partial^2\Omega/\partial T^2$.

Obviously, $T \leq T_c$, where T_c is the critical temperature for condensation and to describe the system we use a version of the Bogoliubov's model [6] valid near T_c, which leads to a spectrum having the abovementioned infrared limit.

2 The Bogoliubov's model near the critical point.

We shall assume an weakly interacting Bose gas under the approximate conditions $a/\lambda \ll 1$ and $na^3 \ll 1$ where a is the scattering length of the two body interaction, $\lambda = h/\sqrt{2\pi mT}$ the thermal wavelength of the particles and n the particle density in the system considered. We will start from the quantized field Hamiltonian expanded in terms of the quantities $a_{\mathbf{p}}^+$, $a_{\mathbf{p}}$ ($\mathbf{p} \neq \mathbf{0}$) as in the usual Bogoliubov's procedure [6]. In what follows we refer briefly to the details of this model by following Pathria [7].

Our case differs from Bogoliubov's one by two facts :1-) as our temperature interval is close to the critical temperature, we cannot consider in all cases that the number of particles in states with $\mathbf{p} \neq 0$ is always much more smaller than n_o, number of particles in the condensate. 2-)The usual approximation $n_o \simeq N$ cannot be made.

Thus we start also from the Hamiltonian of the quantized field for spinless bosons, we assume momentum conservation in the interactions and name $U(\mathbf{r})$ the repulsive potential of the two body interaction, $U_{\mathbf{p1},\mathbf{p2}}^{\mathbf{p'_1},\mathbf{p'_2}} = \frac{1}{V} \int \int d^3\mathbf{r} e^{\frac{i\mathbf{p}\cdot\mathbf{r}}{\hbar}} U(\mathbf{r})$

The scattering length can be written, then, as $a = \frac{mV}{4\pi\hbar^2} U_{\mathbf{p1},\mathbf{p2}}^{\mathbf{p'_1},\mathbf{p'_2}}$, and as we are considering a repulsive potential always $a > 0$. For $\mathbf{p} = 0$ (zero momentum transfer in the colision) $U_o = \int d^3\mathbf{r} U(\mathbf{r})$.

In the temperature interval we are considering the momenta are very small and we can assume that the momentum transfer in each collision is almost zero $\mathbf{p} \simeq 0$. For that reason it is possible to express approximately the matrix elements by using U_o. Then the Hamiltonian is,

$$\hat{H} = \sum \frac{\mathbf{p}^2}{2m} a_{\mathbf{p}}^+ a_{\mathbf{p}} + \frac{U_o}{2V} \sum a_{\mathbf{p'}_1}^+ a_{\mathbf{p'}_2}^+ a_{\mathbf{p2}} a_{\mathbf{p1}}. \tag{3}$$

$a_o a_o \simeq n_o^2$ which allows us to take each one of these creation and annihilation operators as c-numbers proportional to $\sqrt{n_o}$. Then the second term in the Hamiltonian can be expanded, its lowest-order term being $a_o^+ a_o^+ a_o a_o = n_o^2 - n_o$ and the lowest-order ground state energy is $E_o = \frac{U_o}{2V}[n_o^2 - n_o]$. The next order terms to be included leads to the term $\sum_{\mathbf{p} \neq \mathbf{0}} \{ a_{\mathbf{p}}^+ a_{\mathbf{p}}^+ + a_{\mathbf{p}} a_{-\mathbf{p}} + 4 a_{\mathbf{p}}^+ a_{-\mathbf{p}} \} n_o$ and our final Hamiltonian is

$$
\begin{aligned}
\hat{H} &= \frac{U_o}{2V}[n_o^2 - n_o] + \sum_{\mathbf{p} \neq 0} \frac{\mathbf{p}^2}{2m} a_{\mathbf{p}}^+ a_{\mathbf{p}} \\
&+ \frac{U_o n_o}{2V} \sum_{\mathbf{p} \neq 0} \{ a_{\mathbf{p}}^+ a_{-\mathbf{p}}^+ + a_{\mathbf{p}} a_{-\mathbf{p}} + 4 a_{\mathbf{p}}^+ a_{-\mathbf{p}} \} \, .
\end{aligned}
\tag{4}
$$

The usual Bogoliubov's procedure is, by considering that $n_0 \sim N$, (which ceases to be valid near the transition temperature) to replace N by n_0 in the in the third term in (4) (this means to neglect one term of fourth order in the quantities $a_{\mathbf{p}}^+, a_{\mathbf{p}}$), and after introducing the relation $\sum_{\mathbf{p} \neq 0} a_{\mathbf{p}}^+ a_{\mathbf{p}} + n_o = N$ in the first, leads to the cancellation of a term $\frac{U_0 n_0}{V} a_{\mathbf{p}} a_{-\mathbf{p}}^+$ in the last bracket of (4).

Thus, our assumptions lead simply to change the coefficient of the last term in curly brackets from 2 to 4. The next step is to make the usual Bogoliubov transformation $a_{\mathbf{p}} = (b_{\mathbf{p}} - \alpha_{\mathbf{p}} b_{-\mathbf{p}}^+)/\sqrt{1 - \alpha_{\mathbf{p}^2}}$ and $a_{\mathbf{p}}^+ = (b_{\mathbf{p}}^+ - \alpha_{\mathbf{p}} b_{-\mathbf{p}})/\sqrt{1 - \alpha_{\mathbf{p}}^2}$ and as a result of it we get the diagonalized Hamiltonian,

$$
\hat{H} = E_o + \sum_{\mathbf{p} \neq 0} \varepsilon(\mathbf{p}) b_{\mathbf{p}}^+ b_{\mathbf{p}},
\tag{5}
$$

where $E_o = \frac{U_o}{2V}[n_o^2 - n_o] - \frac{U_o n_o}{2V} \sum_{\mathbf{p} \neq 0} \alpha_{\mathbf{p}}$ and $\alpha_{\mathbf{p}} = \frac{V}{U_o n_o}[4\frac{U_o n_o}{2V} + \frac{\mathbf{p}^2}{2m} - \varepsilon(\mathbf{p})]$, and if we name $K = \frac{U_o n_0}{2V}$ then

$$
\varepsilon(\mathbf{p}) = \sqrt{12K^2 + 8K\frac{\mathbf{p}^2}{2m} + (\frac{\mathbf{p}^2}{2m})^2},
\tag{6}
$$

is the spectrum of the new Bose quasiparticles representing the elementary excitations of

In Bogoliubov's spectrum the term $12K^2$ is not present. In our model the long wavelength limit is obviously not linear in p. As $K \ll kT$ is very small (for $n_0 \simeq 10^{16}, K \simeq 10^{-12}$ eV) it can be usually neglected, but it is able to produce the typical λ-type divergent behavior of the specific heat near T_c. As $\lim_{\mathbf{p} \to 0} \varepsilon(\mathbf{p}) = 2\sqrt{3}K$, the parameter K formally behaves as the analog of a rest energy in relativistic dynamics. This "rest energy" has the remarkable property that it decreases with temperature and goes to zero for $T \to T_c$. Actually, K is proportional to the symmetry breaking parameter n_0, which is the condensate, and condensation means some (first order) gauge symmetry breaking [8]. Thus K arises as some sort of non-relativistic analog of the temperature scalar and abelian Higgs models mentioned above. An explicit computation of c_v for this case shows a divergent behavior close to the critical temperature, as it is easily checked by doing the temperature expansion of the thermodynamic potential, which we will do by taking the exact spectrum (6).

The divergent behavior of the specific heat is related to the fact that at the critical temperature a macroscopic number of particles suddenly falls to the ground state. Due to the interaction term, the ground state energy is different from zero,

$$E_o = \frac{2\pi\hbar^2 a n_o^2}{mV} \left\{ 1 + \frac{16}{15} \sqrt{\frac{6a^3 n_o}{\pi V}} \left[7E\left(\frac{\pi}{2}, \sqrt{\frac{2}{3}}\right) - 2F\left(\frac{\pi}{2}, \sqrt{\frac{2}{3}}\right) \right] \right\}, \qquad (7)$$

where E, F are the usual elliptic integrals. Due to interactions there is also a macroscopic number of particles in states very close to the condensate.

3 Quasi-particle thermodynamic potential

The thermodynamic potential of the quasi-particles whose spectrum was obtained in the last section is $\Omega = \frac{V}{\beta\hbar^3} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln[1 - \exp(-\beta\varepsilon(\mathbf{p}))]$ and $\beta = \frac{1}{kT}$. We have taken the chemical potential $\mu = 0$. We are going to do an asymptotic expansion of this potential close to T_c for $T < T_c$, taking into account that for $T \to T_c$, $n_o \to 0$.

$$\Omega = \frac{2(2m)^{3/2}V}{\beta^{5/2}\hbar^3} \int\limits_0^\infty dx x^2 \ln[1 - e^{-\sqrt{12M^2 + 8Mx^2 + x^4}}].,$$

(8)

where $M = K\beta \ll 1$ and we are going to do our expansion in terms of it, in the same way done in the case of the effective potential in the abelian Higgs model [9].

Thus,

$$\Omega = \Omega(M = 0) + \left.\frac{\partial\Omega}{\partial M}\right|_{M=0} M + R(M),$$

(9)

where we stop our expansion in the first-order term and $R(M)$ is certain function of M that we are going to find out as

$$R(M) = \Omega(M) - \Omega(M = 0) - \left.\frac{\partial\Omega}{\partial M}\right|_{M=0} M,$$

(10)

if the first derivative of this expression is calculated we obtain

$$\frac{\partial R}{\partial M} = \frac{\partial\Omega}{\partial M} - = \left.\frac{\partial\Omega}{\partial M}\right|_{M=0},$$

(11)

and this is the expression to be used for computing $R(M)$.

After some calculations that are shown in the appendix the thermodynamic potential is obtained as

$$\Omega(M) = \frac{(2m)^{3/2}}{\hbar^3 (2\pi)^2} \left\{ \frac{-2}{3}\Gamma(\frac{5}{2})\zeta(\frac{5}{2})k^{5/2}T^{5/2} + 4\Gamma(\frac{3}{2})\zeta(\frac{3}{2})k^{3/2}T^{3/2} \right.$$

(12)

$$-\alpha\frac{8\pi}{3}kTK^{3/2}$$

$$-\frac{8}{5}\sqrt{\frac{2}{3}}K^{5/2}[7E(\frac{\pi}{2}, \sqrt{\frac{2}{3}}) - 2F(\frac{\pi}{2}, \sqrt{\frac{2}{3}})]$$

$$\left. +\zeta(\frac{3}{2})k^{1/2}T^{1/2}\frac{O(K^3)}{12\sqrt{\pi^3}} \right\} + const.$$

The most interesting term for us is the third one, containing the infrared contribution. Such behavior can be obtained also by cutting the interval of integration in (8) in the form $\int_0^\infty = \int_0^\eta + \int_\eta^\infty$ by taking some small $\eta \gg M$. Then in the first integral we can expand the exponential in the denominator, leading to a linear in T term, which contains the third term of (13).

4 Internal energy and specific heat

As $K \to 0$ near T_c, in what follows we are taking into account only the terms up to $K^{3/2}$. Then for $U = \Omega - T\frac{\partial\Omega}{\partial T}$ we obtain

$$
\begin{aligned}
U &= \frac{(2m)^{3/2}}{\hbar^3(2\pi)^2}\left\{\Gamma(\frac{5}{2})\zeta(\frac{5}{2})k^{5/2}T^{5/2} - 2\Gamma(\frac{3}{2})\zeta(\frac{3}{2})k^{3/2}T^{3/2}K\right. \\
&\left. -4\Gamma(\frac{3}{2})\zeta(\frac{3}{2})k^{3/2}T^{5/2}\frac{\partial K}{\partial T} + \alpha 4\pi kT^2 K^{1/2}\frac{\partial K}{\partial T}\right\},
\end{aligned}
\tag{13}
$$

and for the specific heat $c_v = \frac{\partial U}{\partial T}$ the expression

$$
\begin{aligned}
c_v = \frac{(2m)^{3/2}}{\hbar^3(2\pi)^2}&\left\{\frac{5}{2}\Gamma(\frac{5}{2})\zeta(\frac{5}{2})k^{5/2}T^{3/2} - 3\Gamma(\frac{3}{2})\zeta(\frac{3}{2})k^{3/2}T^{1/2}K\right. \\[2mm]
&+4kT[2\alpha\pi K^{1/2} - 3\Gamma(\frac{3}{2})\zeta(\frac{3}{2})k^{1/2}T^{1/2}]\frac{\partial K}{\partial T} \\
&-4kT^2[\Gamma(\frac{3}{2})\zeta(\frac{3}{2})k^{1/2}T^{1/2} - \alpha\pi K^{1/2}]\frac{\partial^2 K}{\partial T^2} \\
&\left.+\alpha 2\pi kT^2\left(\frac{\partial K}{\partial T}\right)^2 K^{-1/2}\right\},
\end{aligned}
\tag{14}
$$

and it is the last term in (4) the one giving the divergent behavior of c_v for $T \to T_{c-}$. The first term gives exactly the contribution of the ideal boson gas.

The contribution of the last term can be expressed in terms of n_o as

By taking $n_o = N\left[1 - \left(\frac{T}{T_c}\right)^\gamma\right]$, ($\gamma > 0$, for the ideal gas $\gamma = \frac{2}{3}$), one obtains that for $T \to T_{c-}$, $\triangle c_v \sim \frac{N\gamma^2 T^{2\gamma}}{T^{2\gamma-2c}\sqrt{1-(T/T_c)^\gamma}}$.

The behavior of c_v for $T > T_c$ would require a separate investigation. For exactly $T = T_c$, as argued in a previous paper [10], for the ideal gas not in the thermodynamic limit, the number of particles in the condensate is not zero, and for $T \simeq T_+$ we may expect that the densities in states close to the ground state are large, but rapidly decreasing with increasing T. In that region, the chemical potential cannot be taken as zero. Thus, we expect that a mechanism similar to the one described for $T < T_c$ may perhaps leads to a divergent behavior in that region for $T \to T_c$. We shall take, however, the free gas behavior in our present approximation. We will write also

$$n_o = Nf(T). \tag{16}$$

The specific form of $f(T)$ can be taken as the ideal gas one since usually the quantity $\alpha_{\mathbf{p}} \ll 1$ and thus the average density of quasiparticles $n_{\mathbf{p}}$ is of the same order of the particles not in the ground state $N_{\mathbf{p}}$ (since $N_{\mathbf{p}} = n_{\mathbf{p}}(1 + \alpha_{\mathbf{p}}^2/1 - \alpha_{\mathbf{p}}^2) + \alpha_{\mathbf{p}}^2/1 - \alpha_{\mathbf{p}}^2$), [11], and these are proportional to $T^{3/2}$. We take, thus,

$$f(T) = \left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right], \tag{17}$$

along with helium parameters, the divergent curve $c_v(T)$ for $T = T_c$ shown in figure 1 is obtained. In this case the scattering length has been chosen as the radius of a hard-sphere as $a = 2.14\mathring{A}$ (Kalos,Levesque, and Verlet)[12]. We want to remark that such a divergent behavior cannot be obtained by using the usual Bogoliubov's spectrum [13].

5 Conclusions

From our results we conclude that some quasi-particle boson spectra having nonvanishing temperature-dependent infrared limit, for some specific dependences of this limit on

the weakly interacting gas it is seen that if one does not take the approximation $n_o \simeq N$, but use instead $n_0(T)$, one obtains a quasiparticle spectrum which is nonvanishing in the long wavelength limit, $\lim_{\mathbf{p} \to 0} \varepsilon(\mathbf{p}) = \sqrt{12}K$ where K is proportional to the condensate density n_o. The nonvanishing value of K is a manifestation of the symmetry breaking, and for $n_o = 0$ the spectrum is reduced to the free particle one, where $\lim_{\mathbf{p} \to 0} \varepsilon(\mathbf{p}) = 0$. The infrared contribution to the internal energy contains a term proportional to $n_0^{1/2}$. For a dependence of n_0 on temperature of form $n_0 = N[1 - (T/T_c)^\gamma]$, $\gamma > 0$, a λ-type divergent behavior of the specific heat c_v is obtained as $T \to T_c$.

6   Acknowledgments

7   Appendix

7.1   Lowest and first order terms of Ω

If Ω is evaluated for $M = 0$ the expression for the thermodynamic potential is

$$\Omega(M = 0) = \frac{-2(2m)^{3/2}V}{3\beta^{5/2}\hbar^3(2\pi)^2}\Gamma(\frac{5}{2})\zeta(\frac{5}{2}), \tag{18}$$

Then

$$\frac{\partial\Omega}{\partial M} = \frac{(2m)^{3/2}V}{\beta^{5/2}\hbar^3(2\pi)^2}\int_0^\infty \frac{dx\,x^2[24M + 8x^2]}{\sqrt{12M^2 + 8Mx^2 + x^4}[\exp(\sqrt{12M^2 + 8Mx^2 + x^4}) - 1]},$$

from which

$$\left.\frac{\partial \Omega}{\partial M}\right|_{M=0} = \frac{4(2m)^{3/2}V}{\beta^{5/2}\hbar^3(2\pi)^2}\Gamma(\frac{3}{2})\zeta(\frac{3}{2}). \tag{19}$$

7.2 The $R(M)$ function

If we substitute in (11) the expressions (18) , (19) and the Matsubara sum

$$\frac{1}{\varepsilon[\exp(\varepsilon)-1]} = \sum_{-\infty}^{\infty}\frac{1}{\varepsilon^2+4\pi n^2} - \frac{1}{2\varepsilon}, \tag{20}$$

we obtain

$$
\begin{aligned}
\frac{\partial R}{\partial M} &= \frac{8(2m)^{3/2}V}{\beta^{5/2}\hbar^3(2\pi)^2}\left\{\left[\int_0^{\infty}\frac{dx\,x^2[3M+x^2]}{12M^2+8Mx^2+x^4} - \int_0^{\infty}\frac{dx\,x^4}{x^4}\right]\right.\\
&\quad +2\sum_{n=1}^{\infty}\left[\int_0^{\infty}\frac{dx\,x^2[3M+x^2]}{4\pi n^2+12M^2+8Mx^2+x^4} - \int_0^{\infty}\frac{dx\,x^4}{4\pi n^2+x^4}\right]\\
&\quad \left.-\frac{1}{2}\left[\int_0^{\infty}\frac{dx\,x^2[3M+x^2]}{\sqrt{12M^2+8Mx^2+x^4}} - \int_0^{\infty}\frac{dx\,x^4}{x^2}\right]\right\}.
\end{aligned}
\tag{21}
$$

Each one of the terms between brackets are calculated separately, thus the first one is

$$[...]_0 = \frac{-\alpha\pi}{2}M^{1/2}, \tag{22}$$

where $\alpha = (\frac{5}{4} + \frac{\sqrt{3}}{2})(\sqrt{6}-\sqrt{2})$.

In the second term as M is very small in our range of temperatures we can take the limiting case of n large and expand it in a power series of $\frac{M}{\pi n}$ so

$$[...]_1 = \frac{-3M}{4}\frac{\sqrt{\pi}}{\sqrt{ }} + \frac{1}{4}\frac{M^2}{(\quad)^{3/2}} + ... \tag{23}$$

The last one, after integrating is,

$$[...]_2 = -ML + \sqrt{\frac{2}{3}} M^{3/2} \left\{ 7E(\frac{\pi}{2}, \sqrt{\frac{2}{3}}) - 2F(\frac{\pi}{2}, \sqrt{\frac{2}{3}}) \right\}, \tag{24}$$

where the divergent integrals have been been regularized by introducing a cut-off L large enough.

The first term of $[...]_1$ when the sum over n is done is a divergent series. Thus we take an upper cut-off term at $n = N(L)$ such as to cancel both divergent terms. At the end we get

$$\frac{\partial R}{\partial M} = \frac{8(2m)^{3/2}V}{\beta^{5/2}\hbar^3(2\pi)^2} \left\{ -\alpha\frac{\pi}{2}M^{1/2} + 2\sum_{n=1}^{N}\frac{-3M\sqrt{\pi}}{4\sqrt{n}} + \frac{1}{4}\frac{O(M^2)}{\sqrt{\pi^3}}\zeta(\frac{3}{2}) \right.$$
$$\left. -\frac{1}{2}[-ML + \sqrt{\frac{2}{3}}M^{3/2}(7E(\frac{\pi}{2}, \sqrt{\frac{2}{3}}) - 2F(\frac{\pi}{2}, \sqrt{\frac{2}{3}}))] \right\}, \tag{25}$$

then the sum can be approximated by

$$\sum_{n=1}^{N}\frac{1}{\sqrt{n}} \approx \int_0^N \frac{dn}{\sqrt{n}} = 2N^{1/2}, \tag{26}$$

in this way by taking $L = \frac{6}{\sqrt{\pi}}N^{1/2}$, the otherwise divergent terms cancel each other and after the integration in M has been performed the value of the $R(M)$ function is

$$R(M) = \frac{8(2m)^{3/2}}{\beta^{5/2}\hbar^3(2\pi)^2} \left\{ -\alpha\frac{\pi}{3}M^{3/2} \right.$$
$$-\frac{1}{5}\sqrt{\frac{2}{3}}M^{5/2}[7E(\frac{\pi}{2}, \sqrt{\frac{2}{3}}) - 2F(\frac{\pi}{2}, \sqrt{\frac{2}{3}})]$$
$$\left. +\zeta(\frac{3}{2})\frac{O(M^3)}{12\sqrt{\pi^3}} + const \right\}. \tag{27}$$

where *const* does not depend on M.

References

[2] R.P. Feynman, *Statistical Mechanics* Benjamin, Reading, Mass. (1982).

[3] D.M. Ceperley, *Rev. Mod. Phys.* **67** (1995) 2.

[4] L. Dolan and R. Jackiw, *Phys. Rev.* **D 9** (1974) 3320.

[5] A.D. Linde, *Rep. Prog. Phys* **42** (1979)

[6] N. N. Bogoliubov, *J.Phys.USSR* **11**, (1947) 23.

[7] R. K. Pathria, *Statistical Mechanics*, Pergamon Press (1972).

[8] M. Fannes, J.V. Pule and A. Berbeure, *Helv. Phys. Acta* **55** (1982) 391.

[9] L. Dolan and R. Jackiw, *Phys.Rev.* **D 9**, (1974) 3320.

[10] H. Perez Rojas, *Phys. Let.***A** (to appear)(1997).

[11] L.D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon Press (1966).

[12] M.H. Kalos, D. Levesque and L. Verlet, *Phys.Rev.* **A 9**, (1974) 2178.

[13] K. Huang, in *Bose-Einstein Condensation*, Edited by A. Griffin, D.W. Snoke and S. Stringari, Cambridge University Press (1995).

8   Figure Caption

Figure 1: (c_v, T)−curve obtained from our model, where the divergent behavior of the specific heat is shown. For $T > T_c$ the ideal gas curve has been taken.